

A Combined Solution of the Schwinger-Dyson and Bethe-Salpeter Equations for Mesons as $q\bar{q}$ Bound States

S.M. Dorkin

*Bogoliubov Lab. Theor. Phys. JINR, 141980 Dubna,
Russia and International University Dubna, Russia*

T. Hilger

TU Dresden, Institut für Theoretische Physik, 01062 Dresden, Germany

B. Kämpfer

*Forschungszentrum Dresden-Rossendorf, PF 510119,
01314 Dresden, Germany and TU Dresden,
Institut für Theoretische Physik, 01062 Dresden, Germany*

L.P. Kaptari*

Bogoliubov Lab. Theor. Phys. JINR, 141980 Dubna, Russia

The mass spectrum of heavy pseudoscalar mesons, described as quark-antiquark bound systems, is considered within the Bethe-Salpeter formalism with momentum-dependent masses of the constituents. This dependence is found by solving the Schwinger-Dyson equation for quark propagators in rainbow-ladder approximation. Such an approximation is known to provide both a fast convergence of numerical methods and accurate results for lightest mesons. However, as the meson mass increases, the method becomes less stable and special attention must be devoted to details of numerical means of solving the corresponding equations. We focus on the pseudoscalar sector and show that our numerical scheme describes fairly accurately the π , K , D , D_s and η_c ground states. Excited states are considered as well. Our calculations are directly related to the future physics programme at FAIR.

*supported through the program "Rientro dei Cervelli" of the Italian Ministry of University and Research

I. INTRODUCTION

The description of mesons as quark-antiquark bound states within the framework of the Bethe-Salpeter equation with momentum dependent quark masses, determined by the Schwinger-Dyson equation, is able to explain successfully many spectroscopic data [1–9]. Contrarily to traditional phenomenological models, like quark bag models, the presented formalism maintains important features of QCD, such as dynamical chiral symmetry breaking, dynamical quark dressing, requirements of the renormalization group theory etc. The main ingredients here are the full quark-gluon vertex function and the dressed gluon propagator, the calculation of which is entirely determined by the running coupling and the bare quark masses. In principle, if one were able to solve the Schwinger-Dyson equation within all pQCD orders, the approach would not depend on any free parameters. However, due to known technical problems, one restricts oneself to calculations of the few first terms of the perturbative series, usually up to the one-loop approximation. The obtained results, which formally obey all the fundamental requirements of the theory, are then considered as exact ones with, however, effective parameters. This is known as the rainbow-ladder approximation for the Schwinger-Dyson equation. The merit of the approach is that, once the effective parameters are fixed, the whole spectrum of known mesons is supposed to be described on the same footing. Also the excited states are supposed to be described within this approximation.

It should be noted that there exists other approaches based on the same physical ideas but not so sophisticated, e.g. employing simpler interactions, such as a separable interaction for the effective coupling [6]. Such approaches describe fairly well the properties of light mesons, nevertheless, investigation of heavier mesons and excited states, consisting even of light (u,d,s) quarks, requires implementations of more accurate numerical methods to solve the corresponding equations. Among other successful efforts in this realm the Refs. [10, 11] must be also mentioned.

In the present note we are going to apply the combined Schwinger-Dyson and Bethe-Salpeter (BS) formalisms to describe the meson mass spectrum including heavy mesons and excited states as well. Particular attention is paid to the charm sector which, together with the baryon spectroscopy, is a major topic in the FAIR research programme. Two large collaborations at FAIR [12, 13] plan precision measurements. Note, that it becomes now possible to experimentally investigate not only the mass spectrum of the mentioned mesons, but also different processes of their decay, which are directly connected with fundamental QCD problems (e.g., $U(1)$ axial anomaly, transition form factors etc.) and with the known problem of

changing the meson characteristics at finite temperatures. The latter is crucial in understanding the di-lepton yields in nucleus-nucleus collisions at, e.g. HADES. All these circumstances require an adequate theoretical foundation to describe the meson spectrum and the meson covariant wave functions (i.e. the BS partial amplitudes) needed in calculations of the relevant Feynman diagrams and observables.

II. PROPAGATORS AND SCHWINGER-DYSON EQUATION

The Bethe-Salpeter and Schwinger-Dyson equations in Minkowski space contain poles and branch-point singularities which strongly hinder the procedure of finding numerical solutions. Usually, to avoid these difficulties, one performs the Wick rotation and formulates the corresponding equations in Euclidean space, where all singularities in amplitudes and propagators are removed, so that the equations can be solved numerically. The known Mandelstam technique allows then to calculate matrix elements of observables which, being analytical functions of the relative energy, are the same in both Minkowski and Euclidean spaces.

In our case we consider the Schwinger-Dyson equation for the quark propagator within pQCD with summing all diagrams up to one-loop. In calculations of diagrams the chiral symmetry breaking is implemented from the very beginning [9]. The exact results, even only up to one-loop diagrams, after proper regularization and renormalization procedures are rather cumbersome for further numerical calculations. Nevertheless, in practical calculations one can employ reasonable parametrizations for the corresponding vertices and propagators to find solutions numerically. So, a phenomenological expression, inspired by calculations of the mentioned diagrams and preserving the requirements of the theory, for the combined running coupling and gluon propagator has been suggested in Ref. [9]

$$g^2(k^2)D_{\mu\nu}(k^2) = \left(\frac{4\pi^2 D k^2}{\omega^2} e^{-k^2/\omega^2} + \frac{8\pi^2 \gamma_m F(k^2)}{\ln[\tau + (1 + \frac{k^2}{\Lambda_{QCD}^2})]} \right) \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right), \quad (1)$$

where the first term originates from the infrared (IR) part of the interaction determined by non-perturbative effects, while the second one ensures the correct ultraviolet asymptotics. According to the known fact that the contribution of the IR part is predominant for formation of bound states, in what follows we neglect the second term and restrict ourselves to the IR one. Then, as seen from Eq. (1), we are left with only two effective parameters, D and ω .

The calculation of the renormalized Feynman diagrams leads to a fermion propagator depending on two additional parameters. In canonical calculations these are the renormalization

constant Z_2 and the self-energy corrections $\Sigma(p)$. Usually, for further simplifications of calculations, instead of Z_2 and $\Sigma(p)$ one introduces other two quantities $A(p)$ and $B(p)$ in terms of which the quark propagator $S_q(p)$ reads as

$$S_q^{-1}(p) = i\gamma \cdot p A(p) + B(p); \quad S_q(p) = \frac{-i\gamma \cdot p A(p) + B(p)}{p^2 A^2(p) + B^2(p)}. \quad (2)$$

Then with such a representation of the quark propagator the Schwinger-Dyson equation in Euclidean space has the form (cf. Refs. [7, 9])

$$S_q^{-1}(p) = i\gamma \cdot p + \tilde{m} + \frac{4}{3} \int \frac{d^4 l}{(2\pi)^4} [g^2 D(p-l)]_{\mu\nu} \gamma_\mu S_q(l) \gamma_\nu, \quad (3)$$

where \tilde{m} is the bare quark mass and the effective kernel $D(p-l)$ is

$$D(k^2)_{\mu\nu} = \frac{4\pi^2 D k^2}{\omega^2} e^{-k^2/\omega^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right). \quad (4)$$

Note that (3) is a four dimensional integral equation. To solve it one usually decomposes the kernel over a complete set of basis functions, performs analytically some angular integrations and considers a new system of equations relative to such a partial decomposition. In our calculations we expand the interaction kernel into hyperspherical harmonics

$$D((p-l)^2) = D(p^2, l^2, |p||l| \cos \gamma) = \sum_n D_n(p, l) C_n^1(\cos \gamma), \quad (5)$$

where $C_n^1(t)$ are the Chebyshev polynomials and $D'_n(p, l)$ denotes the corresponding part of the kernel $D(k^2)/k^2$. Eventually, the system of equations to be solved reads as

$$\begin{aligned} A(p) &= 1 + \frac{4}{3p} \int \frac{dl l^4}{16\pi^2} \frac{A(l)}{l^2 A^2(l) + B^2(l)} D_1(p, l) + \\ &\quad \int \frac{dl l^4}{16\pi^2} \frac{A(l)}{l^2 A^2 + B^2} \left[\frac{8}{3} \left(p + \frac{l^2}{p} \right) D'_1(p, l) - \frac{4}{3} l (D'_2(p, l) + 5D'_0(p, l)) \right], \\ B(p) &= \tilde{m} + 4 \int \frac{dl l^3}{8\pi^2} \frac{B(l)}{l^2 A^2(l) + B^2(l)} D_0(p, l). \end{aligned} \quad (6)$$

The resulting system of equations on (6) is a system of one-dimensional integrals and can be solved numerically, e.g. by an iteration method. We found that iteration procedure for (6) converges rather fast and practically does not depend up on the choice of the trial functions for $A(p)$ and $B(p)$.

III. BETHE-SALPETER VERTEX FUNCTION

To determine the bound state mass of a quark-antiquark pair one needs to solve the Bethe-Salpeter equation. In rainbow-ladder approximation and in Euclidean space it reads [7, 9]

$$\Gamma(P, p) = \left(-\frac{4}{3}\right) \int \frac{d^4 k}{(2\pi)^4} \gamma_\mu S(k_+) \Gamma(P, k) S(k_-) \gamma_\nu (g^2 D(p-k))_{\mu\nu}, \quad (7)$$

with Γ being the Bethe-Salpeter vertex function. The color structure has been factorized explicitly. P means the total momentum of the bound state, $p(k)$ is the relative momentum within the quark pair and $k_+ = k + \xi P$, $k_- = k + (\xi - 1)P$. The result does not depend on ξ due to the covariance of the Bethe-Salpeter equation.

Equation (7) is written in matrix form, i.e. the vertex function $\Gamma(P, k)$ is a 4x4 matrix and, therefore, may contain 16 different functions. The general structure of the vertex functions describing bound states of spinor particles has been investigated in detail, for example, in [14]. Γ can be expanded into spin - angular momentum functions:

$$\Gamma(p_4, \mathbf{p}) = \sum_{\alpha} g_{\alpha}(p_4, |\mathbf{p}|) \mathcal{T}_{\alpha}(\mathbf{p}) \quad (8)$$

and in functions $g_{\alpha}(p_4, |\mathbf{p}|)$ in its turn the dependence on the angle χ can be written as

$$g_i(p_4, |\mathbf{p}|) = \sum_j g_i^j(p) X_{j,0(1)}(\chi_p). \quad (9)$$

This allows to perform the angular integration leaving us with a set of coupled one-dimensional linear integral equations for the functions $g_i^k(p)$

$$g_i^k(p) = \int_0^{\infty} \frac{dk k^3}{4\pi^2} \sum A_{ij}^{km}(p, k) g_j^m(k) \quad (10)$$

where $p(k)$ is the length of the Euclidean 4-momentum, $p = \sqrt{p_4^2 + \mathbf{p}^2}$. The matrices $A_{ij}^{km}(p, k)$ can be calculated analytically.

The series (9) converges rather fast and in practice only few terms need to be taken into account. Then by choosing a suitable method of integration, e.g. Gaussian quadrature, the system of equations (10) can be written as a system of homogeneous linear equations. Schematically, it can be written in the form

$$g = K g. \quad (11)$$

The condition $\det(K - 1) = 0$ is sufficient for the existence of a bound state. Hence, formally the zeros of the determinant $\det(K - 1)$ determine the solution of the BS equation, including also excited states.

Here an important moment is worth to be emphasized. Usually, when solving the BS equation for constituent particles, i.e. for particles with constant masses, the resulting system of partial equations is real. In case of momentum dependent masses the Bethe-Salpeter equation becomes complex and requires the knowledge of the quark propagator for complex momenta k_{\pm} (where $k_{\pm} = \frac{1}{2}P \pm k$ are the momenta of quarks) and, hence, requires to solve the

Schwinger-Dyson equation also for complex momenta. For small meson masses (e.g. $M \approx 500$ MeV) only a small energy region contributes to the integral in Eq. (10), $k < 1$ GeV. In this case the propagator functions can be obtained by using the solution for real momenta, and afterwards $A(p)$ and $B(p)$ are calculated at complex momenta from (6) [3, 7]. For heavier states, $M > 1$ GeV, the imaginary part of the quark momenta, $\text{Im } k_{\pm}$, becomes rather large and the integrand in (6) rapidly oscillates as a function of k , hindering an accurate computation of the integral.

To avoid this problem a witty trick has been suggested in Ref. [8]. It is based on the observation that, as seen from the Schwinger-Dyson equation, all the values of the momenta k_{\pm}^2 are located within a domain limited by a parabola.

Then, due to the Cauchy's theorem, it suffices to know the solution along this parabola to be able to compute it everywhere inside the corresponding domain [16].

IV. NUMERICAL RESULTS

As an example of our numerical study we exhibit in Fig. 1 the energy of the lowest bound states of a hypothetical meson qq_x consisting of one given quark q with the mass known from the Schwinger-Dyson equation, bound with a second quark q_x for which the input bare mass \tilde{m}_x is let to vary arbitrarily. The corresponding effective parameters have been chosen as $\omega = 0.5$ GeV and $D = 16$ GeV⁻² [7, 9] and the bare masses for q correspond to (u, d, s) quarks, $q=u$ (with $\tilde{m}_u = 0.005$ GeV), $q=s$ (with $\tilde{m}_s = 0.115$ GeV) and $q=c$ (with $\tilde{m}_c = 1.0$ GeV). This figure illustrates the whole mass spectrum of pseudoscalar mesons with masses up to $3 \text{ GeV}/c^2$. So, if the q_x quark corresponds to a c quark, then at the intersection of the vertical line $\tilde{m}_x = m_c$ ($\approx 1 \text{ GeV}/c^2$) with "ux" curve one obtains the D -meson (with the quark contents auc), with the "sx" curve the D_s meson and with the "cx" curve - the η_c meson, respectively. It is worth noting that the "ux" curve crosses the $\tilde{m}_u = 0.115$ GeV line roughly at the same value of M_{qq_x} as the "sx" curve crosses the $\tilde{m}_u = 0.005$ GeV line, thus proving a check of consistency of the approach and, at the same time, describing correctly the lowest pseudoscalar us state corresponding to the K meson. It can be seen that even without a fine tuning of $\tilde{m}_{s,c}$ the meson mass spectrum is reproduced fairly well: 135 MeV (π^0 meson), 497 MeV (K meson), 1870 MeV (D^{\pm} meson), 1970 MeV (D_s^{\pm} meson) and 2980 MeV (η_c meson).

Clearly, a special parameterization of the gluon propagators with two adjusted quantities and the self consistent determination of three bare quark masses serve as input for obtaining

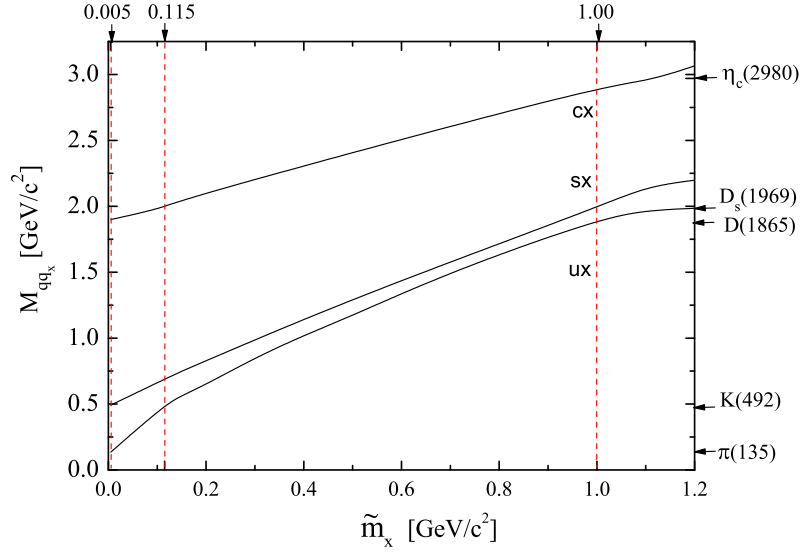


FIG. 1: The bound state masses of a system qq_x as a function of the bare mass \tilde{m}_x , the other quark being either a u quark with $\tilde{m}_u = 0.005$ GeV (the curve labeled as " ux ") or a strange quark with $\tilde{m}_s = 0.115$ GeV (the curve labeled as " sx ") or a charm quark with $\tilde{m}_c = 1.0$ GeV (the curve labeled as " cx ").

finally five pseudoscalar meson masses. The straight forward extension to the scalar, vector and axialvector mesons without further adjustments will provide the ultimate test for the subtleties of the numerical implementation of the employed theoretical scheme and the inherent approximations. Work along this line is in progress.

V. CONCLUSION

The method of solving the Schwinger-Dyson equation in rainbow-ladder approximation in Euclidean space by using the hyperspherical harmonics basis is proposed. The obtained numerical solutions are then used to solve the Bethe-Salpeter equation for the meson mass spectrum in a large interval of meson masses. In solving the Bethe-Salpeter equation a new set of basis functions has been used which allows a further easy decomposition of the Bethe-Salpeter vertex functions into hyperspherical harmonics basis

The obtained mass spectrum for pseudoscalar mesons in a wide range, ranging from pions to η_c mesons, is in a good agreement with experimental data. Excited states were considered as well and found to be also in a good agreement with experimental data and with calculations

by other groups. By solving the Bethe-Salpeter equation, the corresponding partial wave functions are also obtained which will allow, in future, to calculate a variety of observables related to physical programmes at, e.g. FAIR.

Acknowledgments

This work was supported in part by the Heisenberg - Landau program of the JINR - FRG collaboration, GSI-FE and BMBF 06DR9059. Calculations were partially performed on Caspur facilities under the Standard HPC 2010 grant "SRCnuc".

-
- [1] Jain P., Munczek H.J.: Phys. Rev. D **44**, p. 1873, 1991
 - [2] Munczek H.J., Jain P.: Phys. Rev D **46**, p. 438, 1992
 - [3] Maris P., Roberts C.D.: Phys. Rev. C **56**, p. 3369, 1997
 - [4] Frank M.R., Roberts C.D.: Phys. Rev. C **53**, p. 390-398, 1996
 - [5] Tandy P.: Prog. Part. Nucl. Phys. **39**, p. 117, 1997
 - [6] Horvatic D., Blaschke D., Klabucar D., Radzabov A.E.: Phys. Part. Nucl. **39**, p. 1033-1039, 2008
 - [7] Alkofer R., Watson P., Wiegel H.: Phys. Rev. D **65**, 094026, 2002
 - [8] Fisher C.S., Watson P., Cassing W.: Phys. Rev. D **72**, p. 094025, 2005
 - [9] Roberts C.D., Bhagwat V.S., Wright S.V., Holl A.: Eur. Phys. J. ST **140**, p. 53-116, 2007
 - [10] Krassnigg A.: Phys. Rev. D **80**, 114010, 2009
 - [11] Souchlas N.: arXiv:1006.0942v1 [nucl-th]
 - [12] CBM Collaboration, http://www.gsi.de/fair/experiments/CBM/index_e.html.
 - [13] PANDA Collaboration, http://www-panda.gsi.de/auto/phy/_home.htm.
 - [14] Kubis J.J.: Phys. Rev D **6**, p. 547, 1972
 - [15] Dorkin S.M., Beyer M., Semikh S.S., Kaptari L.P.: Few Body Syst. **42**, p. 1-32, 2008
 - [16] Ioakimidis N.I., Papadakis K.E., Perdios E.A.: BIT **31**, p. 276, 1991
 - [17] Nakamura K. et al., J. Phys. G **37**, 075021, 2010